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Local state probability of higher-spin CSOS models

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Abstract. An exact solution of the local state probability (spontaneous magnetization) of a fusion cyclic solid-on-solid (CSOS) model is obtained for a special (2-fusion) case. It is composed of ratios of theta functions and branching coefficients. Using the modular property its critical behaviour is analysed. In addition, a conjecture for the local state probability of the N-fusion CSOS model is proposed.

1. Introduction

Local state probabilities (LSPs) of many 2D lattice models are modular functions. Considering the reason why the spontaneous magnetization of the 2D Ising model is obtained exactly [1], one notices that there is a hidden structure of modular functions. This structure is more explicitly exposed [2] for the eight vertex solid-on-solid (8VSOS) model. The 8VSOS model is a generalization of the 2D Ising model and the hard hexagon model, satisfying the Yang–Baxter equation, and is exactly solved by the corner transfer matrix method. The state variable takes $1, 2, \ldots, L-1(\geqslant 3)$. Its Boltzmann weight is parametrized by elliptic functions with nome p. The final output LSP is expressed in a q-series of the conjugate nome q.

Subsequently we only consider the so-called regime III cases [2]. This regime corresponds to an ordered phase between $T = 0$ and $T = T_c$. After the corner transfer matrix calculation the LSP is given in a low-temperature expansion form. At this stage we cannot see the critical behaviour of the system since all terms in the expansion contribute at the critical point. However if we can sum the infinite series as a modular function or a combination of modular functions, we can see the critical behaviour of the system by a conjugate modulus transformation.

The LSP of 8VSOS model was later shaped in compact forms of affine Lie algebra characters (and their branching coefficients) [3]. The characters are ratios of theta functions. The branching coefficients are expansion coefficients in the theta-function sums-of-products identities [3]. For higher-spin extensions (fusion model) of the 8VSOS model the LSPs have the same structure [3], i.e. they comprise affine Lie algebra characters and branching coefficients. We shall refer to this model as the spin-$N/2$ restricted solid-on-solid (RSOS) model.

Recently we have presented a new hierarchy for solvable models, the higher-spin CSOS model [4]. We constructed this model by the fusion procedure from the $A^{(1)}_{L-1}$ model in [5] or from the CSOS model in [6].

As we have pointed out there, the spin-$N/2$ CSOS model can be constructed if $N \leqslant L-1$ ($N \leqslant L/2 - 1$) for odd (even) L. In this paper we aim to calculate the

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LSP of this higher-spin CSOS model. Although the method used in this paper can be applied to N-fusion models with arbitrary N, we shall concentrate on the calculations for the 2-fusion case specifically. The main part of the calculation is a summation over paths. We expect the resulting expression to be in terms of theta functions. We make the calculation into a recursion formula and iterate the recursion relation to a sufficiently large order. Then we compare the calculated result with the expansion of the expected expression. We use REDUCE for the calculations.

In section 2 we briefly explain how one can calculate the LSPs of these models by the corner transfer matrix method. The techniques shown here are applied to our CSOS model in section 3. In particular we shall fully analyse the spin 1 (2-fusion) CSOS model and obtain a recursion relation. In section 4 we evaluate the solution of this recursion relation in the thermodynamic limit, and obtain the LSP in terms of modular functions. In section 5 we give some concluding remarks.

Here we shall give some notation which will be used later. We use two nomes q and $t(= p^{L/2})$ $(0 \leqslant q,\ t \leqslant 1)$, they are conjugate to each other $((\log q)(\log t) = 4\pi^2)$. We parameterize the nomes as follows,

$$q = \mathrm{e}^{-8\pi^2/\varepsilon} \qquad t = \mathrm{e}^{-\varepsilon/2} \tag{1.1}$$

with a real parameter ε. The nome q is used for a low-temperature parameter, while the nome t measures the deviation from the critical point. We also use the following notation

$$x = \mathrm{e}^{-4\pi^2/\varepsilon}(= \sqrt{q}) \qquad w = \mathrm{e}^{-4\pi^2 u/\varepsilon}. \tag{1.2}$$

We use the parameter w as a label that characterizes the double limit, $\varepsilon \to 0$ and simultaneously $u \to 0$, with a fixed value of u/ε. The former corresponds to the low temperature limit, while the latter means the extreme anisotropy limit.

2. Calculation of the LSP from the Boltzmann weight

In this section we shall briefly summarize the LSP calculation by the corner transfer matrix method, that is used in the next section.

Define $P(a|b,c)$ as the probability, in classical statistical mechanics, for a particular site having the spin value a when a background configuration is specified by b, c. This P is the LSP. A background configuration b, c is a type of boundary condition for an infinitely extended lattice system on which the checkerboard b, c configuration is realized at the low-temperature limit. (We only consider the case where a and b are on the same sublattice.) In later sections we also use $r \equiv \frac{1}{2}(b+c-N)$, $s \equiv \frac{1}{2}(b-c+N)+1$, in order to label a background configuration for N-fusion (spin-$N/2$) models.

We calculate the LSP by using the corner transfer matrix method. From local Boltzmann weights, parametrized by elliptic theta functions and satisfying the Yang–Baxter equation (this is crucial for the following calculation techniques, see [2, 7]), we construct a corner transfer matrix. Due to the Yang–Baxter equation and therefore the commuting row-to-row transfer matrices, the corner transfer matrix becomes an exponential function with respect to the spectral parameter u in the thermodynamic limit. We can diagonalize the matrix in a non-trivial way by taking a double limit of two parameters, the spectral parameter u and the elliptic nome $p = e^{-\varepsilon/L}$.

$$u \to -0 \qquad \varepsilon \to +0, (u/\varepsilon): \text{ fixed}. \tag{2.1}$$

This diagonalization arises from the following diagonal property of the local Boltzmann weight S in the double limit (2.1),

$$\lim_{\substack{\varepsilon\to 0, u\to 0 \\ u/\varepsilon:\text{fixed}}} S\left(\begin{matrix} a & d \\ b & c \end{matrix}\middle| u\right) = \delta_{b,d} \cdot w^{-H(a,b,c)} \tag{2.2}$$

where $w = x^u$. In general a matrix element of a corner transfer matrix is constructed as a summation over infinitely many internal spin configuraions, so it has infinitely many terms. However by this diagonalization all but one term should vanish for each of the diagonal elements. In this diagonalized corner transfer matrix, its diagonal elements are labelled by admissible paths $(\sigma_1, \sigma_2, \ldots)$ of the model. If the system is infinitely large, they are exactly the eigenvalues of the matrix even for finite values of u and ε. Let $A(u)$, $B(u)$, $C(u)$ and $D(u)$ be the four corner transfer matrices corresponding to the four quadrants of the 2D space as usual [7]. We assume that they are given to a specified background configuration b, c. Because of the symmetries in the local Boltzmann weights, $C(u) = A(u)$ and $B(u) = D(u) = A(-1-u)$ up to the gauge factors. We take a partial trace as

$$\mu(a|b,c) = \mathrm{Tr}_{\{\sigma_1=a\}}\, A(u)B(u)C(u)D(u) \tag{2.3}$$

where the spin value at the central site is fixed to a. Having evaluated this trace, we can get the LSP as

$$P(a|b,c) = \frac{\mu(a|b,c)}{\sum_{a'} \mu(a'|b,c)}. \tag{2.4}$$

The partial trace $\mu(a|b,c)$ is a 1D configuration sum multiplied by gauge factors. First we give the finite sum X_m over finite step paths.

$$S_m(\sigma_1, \ldots, \sigma_{m+2}) = \sum_{j=1}^{m} j \cdot H(\sigma_j, \sigma_{j+1}, \sigma_{j+2}) \tag{2.5}$$

$$X_m(a|b,c) = \sum_{\substack{(\sigma_1,\ldots,\sigma_{m+2})\in\text{paths} \\ \sigma_1=a, \sigma_{m+1}=b, \sigma_{m+2}=c}} q^{S_m(\sigma_1,\ldots,\sigma_{m+2})}. \tag{2.6}$$

As $x^u \cdot x^{-1-u} \cdot x^u \cdot x^{-1-u} = x^{-2}$, the factor $w^{-H(\bullet,\bullet,\bullet)}$ appearing in (2.2) has been converted to $q^{H(\bullet,\bullet,\bullet)}$. Afterwards we take the limit $m \to \infty$ to obtain the true trace of the corner transfer matrix.

The summation over paths means a particular combinatorial summation for each model, for instance restricted paths for the RSOS model and cyclically restricted paths for the CSOS model. The difference between the LSP (therefore the physics) of the models arises from the difference between the sets of these paths and between the weight function $H(\bullet, \bullet, \bullet)$ in (2.2).

3. LSP calculation for fusion CSOS models

Let us consider the LSP calculation for higher-spin CSOS models. For spin-$N/2$ RSOS models the weight function H is simply $H(a,b,c) = |a-c|/4$ for all N [3]. But for spin-$N/2$ CSOS models, we find that the behaviour of the weight function H should depend on N. Due to the periodic property of the weights, spin configurations around a face that have such spin values as 0 or $(L-1)$ give exceptional weight function values.

Subsequently we consider the spin-1 CSOS model exclusively. To make this paper self-contained we shall write its Boltzmann weights as in [4].

$$S_{22}\left(\begin{matrix} a & a \\ a & a \end{matrix}\middle| u\right) = \frac{\Theta(\omega_0+a+u)\Theta(\omega_0+a-1-u)}{\Theta(\omega_0+a)\Theta(\omega_0+a-1)} + \frac{\Theta(\omega_0+a+2)\Theta(\omega_0+a-1)H(1+u)H(u)}{\Theta(\omega_0+a+1)\Theta(\omega_0+a)H(2)H(1)}$$

$$S_{22}\left(\begin{matrix} a & a\pm 2 \\ a\pm 2 & a \end{matrix}\middle| u\right) = \frac{\Theta(\omega_0+a\pm 1\mp u)\Theta(\omega_0+a\mp u)}{\Theta(\omega_0+a\pm 1)\Theta(\omega_0+a)}$$

$$S_{22}\left(\begin{matrix} a & a\pm 2 \\ a\mp 2 & a \end{matrix}\middle| u\right) = \frac{H(u)H(-1+u)}{H(2)H(1)}\frac{\sqrt{\Theta(\omega_0+a+2)\Theta(\omega_0+a-2)}}{\Theta(\omega_0+a)}$$

$$S_{22}\left(\begin{matrix} a & a\pm 2 \\ a & a \end{matrix}\middle| u\right) = S_{22}\left(\begin{matrix} a & a \\ a\pm 2 & a \end{matrix}\middle| u\right) = \frac{H(u)\Theta(\omega_0+a\mp u)}{H(1)\Theta(\omega_0+a\pm 1)}\sqrt{\frac{\Theta(\omega_0+a\pm 2)}{\Theta(\omega_0+a)}}$$

$$S_{22}\left(\begin{matrix} a & a \\ a & a\pm 2 \end{matrix}\middle| u\right) = \frac{H(1+u)\Theta(\omega_0+a\pm 1\pm u)}{H(1)\Theta(\omega_0+a\pm 1)}$$

$$S_{22}\left(\begin{matrix} a & a\pm 2 \\ a & a\pm 2 \end{matrix}\middle| u\right) = S_{22}\left(\begin{matrix} a & a \\ a\pm 2 & a\pm 2 \end{matrix}\middle| u\right) = \frac{H(1+u)H(u)}{H(2)H(1)}\frac{\sqrt{\Theta(\omega_0+a\pm 3)\Theta(\omega_0+a\mp 1)}}{\Theta(\omega_0+a\pm 1)}$$

$$S_{22}\left(\begin{matrix} a & a\pm 2 \\ a\pm 2 & a\pm 2 \end{matrix}\middle| u\right) = \frac{H(1+u)\Theta(\omega_0+a\pm 1\mp u)}{H(1)\Theta(\omega_0+a\pm 1)}$$

$$S_{22}\left(\begin{matrix} a & a\pm 2 \\ a\pm 2 & a\pm 4 \end{matrix}\middle| u\right) = \frac{H(2+u)H(1+u)}{H(2)H(1)} \tag{3.1}$$

where $H(v) = \vartheta_1(\pi v/L, t^{2/L})$ and $\Theta(v) = \vartheta_4(\pi v/L, t^{2/L})$.

We use the following notation for the elliptic theta functions,

$$\vartheta_1(v,p) = 2p^{1/8}\sin v \prod_{n=1}^{\infty}(1-2p^n\cos 2v+p^{2n})(1-p^n) \tag{3.2}$$

$$\vartheta_4(v,p) = \prod_{n=1}^{\infty}(1-2p^{n-1/2}\cos 2v+p^{2n-1})(1-p^n). \tag{3.3}$$

These Boltzmann weights have the following symmetries

$$\begin{aligned} S_{22}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) &= S_{22}\left(\begin{matrix} d & a \\ c & b \end{matrix}\middle| u\right) = S_{22}\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right) \\ &= S_{22}\left(\begin{matrix} a & d \\ b & c \end{matrix}\middle| -1-u\right) \times \frac{g_a g_c}{g_b g_d} \end{aligned} \tag{3.4}$$

where $g_l = \varepsilon_l\sqrt{\Theta(l+\omega_0)}$ ($\varepsilon_l = \pm 1$, $\varepsilon_l\varepsilon_{l+1} = (-)^l$). Let us denote by $A(u)$, $B(u)$, $C(u)$ and $D(u)$ the corner transfer matrices with respect to the SE, NE, NW and SW quadrants of the 2D space, constructed from S_{22}. Provided that the central spin value equals a, they satisfy $C(u) = A(u)$ and $B(u) = D(u) = g_a A(-1-u)$ up to a-independent factors because of (3.4).

Next let us replace the constituent Boltzmann weight S_{22} of $A(u)$ as follows

$$S_{22}\left(\begin{matrix} a & d \\ b & c \end{matrix}\middle| u\right) = \mathcal{S}\left(\begin{matrix} a & d \\ b & c \end{matrix}\middle| u\right) \times \frac{G_a(u)G_c(u)}{G_b(u)G_d(u)} x^{u(u+1)/L} \tag{3.5}$$

where $x = \sqrt{q}$ and the spectral parameter-dependent gauge factor $G_l(u)$ is

$$G_l(u) = (x^u)^{(L/4)\cdot W((l+\omega_0)/L)} \qquad W(v) \equiv (v-\lfloor v\rfloor)(v-\lfloor v\rfloor-1). \tag{3.6}$$

This transformed Boltzmann weight $\mathcal{S}$ also satisfies the Yang–Baxter equation and has a suitable form for the double limit (2.2). We are to obtain the weight function $H(\bullet,\bullet,\bullet)$ from this $\mathcal{S}$, and to consider the recursion relation of X_m given by this weight function. Denoting the SE quadrant corner transfer matrix constructed from $\mathcal{S}$ by $\mathcal{A}(u)$, one can see that $A(u) = \mathcal{A}(u)G_a(u)$ up to a-independent factors. In the thermodynamic limit the eigenvalues of this $\mathcal{A}(u)$ are labelled by the admissible paths of the model $(\sigma_1,\sigma_2,\ldots)$ and have such forms as $(x^u)^{\sum_{j=1}^{\infty} j\cdot H(\sigma_j,\sigma_{j+1},\sigma_{j+2})}$. Putting all these together we have

$$\begin{aligned} &A(u)B(u)C(u)D(u) \\ &\quad = G_a(u)^2 \times G_a(-1-u)^2 \times g_a^2 \times \mathcal{A}(u)\mathcal{A}(-1-u)\mathcal{A}(u)\mathcal{A}(-1-u) \end{aligned} \tag{3.7}$$

up to a-independent factors. With respect to the jth term of (2.5) we have

$$((x^u)^{j\cdot H(\sigma_j,\sigma_{j+1},\sigma_{j+2})})^2((x^{-1-u})^{j\cdot H(\sigma_j,\sigma_{j+1},\sigma_{j+2})})^2 = q^{j\cdot H(\sigma_j,\sigma_{j+1},\sigma_{j+2})} \tag{3.8}$$

and since

$$G_a(u)^2 \times G_a(-1-u)^2 = (x^u)^{(a(a-L)/4L)2} \times (x^{-1-u})^{(a(a-L)/4L)2} = x^{-a(a-L)/2L} \tag{3.9}$$

the trace (3.7) and as a result the LSP does not depend on the spectral parameter u as usual.

Next we give the result for the weight function H of the model. As an example let us first show that for this model we have

$$H(0,0,0) = H(L-1,L-1,L-1) = 1. \tag{3.10}$$

Since no gauge factor is needed for this configuration, we have

$$\mathcal{S}\left(\begin{matrix} a & a \\ a & a \end{matrix}\middle| u\right) = \left[\frac{\Theta(\omega_0+a+u)\Theta(\omega_0+a-1-u)}{\Theta(\omega_0+a)\Theta(\omega_0+a-1)} + \frac{\Theta(\omega_0+a+2)\Theta(\omega_0+a-1)H(1+u)H(u)}{\Theta(\omega_0+a+1)\Theta(\omega_0+a)H(2)H(1)}\right] x^{-u(u+1)/L}. \tag{3.11}$$

Assuming $0 < \omega_0 < 1$ and taking the double limit (2.2) of the Boltzmann weight (3.11) we have

$$\text{(the first term)} \to \begin{cases} w^{-1} & \text{for } a = 0 \\ 1 & \text{otherwise} \end{cases}$$
$$\text{(the second term)} \to \begin{cases} w^{-1}-1 & \text{for } a = L-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Therefore we get (3.10).

We also evaluate the double limit (2.2) and obtain the weight function H for all other configurations. We find that they are indeed diagonal if $0 < \omega_0 < 1$. The result is as follows.

$$H(a,b,c) = |a-c|/4$$

if

$$\begin{aligned}(a,b,c) \neq {} & (3,1,L-1),(L-1,1,3),(2,0,L-2),(L-2,0,2),\\ & (1,L-1,L-3),(L-3,L-1,1),(0,L-2,L-4),\\ & (L-4,L-2,0),(0,0,0),(L-1,L-1,L-1),\\ & (0,L-2,0),(1,L-1,1),(L-1,1,1),\\ & (1,1,L-1),(L-2,0,0),(0,0,L-2),\\ & (L-1,L-1,1),(1,L-1,L-1),(L-2,L-2,0),\\ & (0,L-2,L-2),(L-2,0,L-2),(L-1,1,L-1)\end{aligned}$$

$$\begin{aligned}
& H(2,0,L-2) = H(1,1,L-1) = \omega_0/2 \\
& H(3,1,L-1) = H(0,0,L-2) = 1/2+\omega_0/2 \\
& H(1,L-1,L-3) = H(0,L-2,L-2) = 1/2-\omega_0/2 \\
& H(0,L-2,L-4) = H(1,L-1,L-1) = 1-\omega_0/2 \\
& H(0,L-2,0) = H(1,L-1,1) = 1-\omega_0 \\
& H(L-2,0,L-2) = H(L-1,1,L-1) = \omega_0 \\
& H(a,b,c) = H(c,b,a).
\end{aligned} \tag{3.13}$$

We are considering admissible paths $(\sigma_1, \sigma_2, \ldots)$ which satisfy $|\sigma_i - \sigma_{i+1}| = 0, 2 \pmod{L}$ on the space with a periodic boundary ($0 \equiv L$). This weight function $H(\bullet, \bullet, \bullet)$ gives exceptional values for those passages $(\sigma_j, \sigma_{j+1}, \sigma_{j+2})$ which cross or lie on the 'formal boundaries' (0 and $L-1$) of the space (figure 1). This shows that $(L-1, L-1), (0,0), (L-1, 1), (1, L-1), (L-2, 0)$ and $(0, L-2)$ are not allowed to be background configurations. Using (3.13) we can write the recursion relation of X_m for this model. For simplicity we assume $\omega_0 \to +0$.

$$\begin{pmatrix} X_m(a|b,b+2) \\ X_m(a|b,b) \\ X_m(a|b,b-2) \end{pmatrix} = \begin{pmatrix} 1 & q^{m/2} & q^m \\ q^{m/2} & 1 & q^{m/2} \\ q^m & q^{m/2} & 1 \end{pmatrix} \begin{pmatrix} X_{m-1}(a|b+2,b) \\ X_{m-1}(a|b,b) \\ X_{m-1}(a|b-2,b) \end{pmatrix}$$

if $b \neq 1, 0, L-1, L-2$

$$\begin{pmatrix} X_m(a|1,3) \\ X_m(a|1,1) \\ X_m(a|1,L-1) \end{pmatrix} = \begin{pmatrix} 1 & q^{m/2} & q^{m/2} \\ q^{m/2} & 1 & 1 \\ q^{m/2} & 1 & 1 \end{pmatrix} \begin{pmatrix} X_{m-1}(a|3,1) \\ X_{m-1}(a|1,1) \\ X_{m-1}(a|L-1,1) \end{pmatrix}$$

$$\begin{pmatrix} X_m(a|0,2) \\ X_m(a|0,0) \\ X_m(a|0,L-2) \end{pmatrix} = \begin{pmatrix} 1 & q^{m/2} & 1 \\ q^{m/2} & q^m & q^{m/2} \\ 1 & q^{m/2} & 1 \end{pmatrix} \begin{pmatrix} X_{m-1}(a|2,0) \\ X_{m-1}(a|0,0) \\ X_{m-1}(a|L-2,0) \end{pmatrix}$$

$$\begin{pmatrix} X_m(a|L-1,1) \\ X_m(a|L-1,L-1) \\ X_m(a|L-1,L-3) \end{pmatrix} = \begin{pmatrix} q^m & q^m & q^{m/2} \\ q^m & q^m & q^{m/2} \\ q^{m/2} & q^{m/2} & 1 \end{pmatrix} \begin{pmatrix} X_{m-1}(a|1,L-1) \\ X_{m-1}(a|L-1,L-1) \\ X_{m-1}(a|L-3,L-1) \end{pmatrix}$$

$$\begin{pmatrix} X_m(a|L-2,0) \\ X_m(a|L-2,L-2) \\ X_m(a|L-2,L-4) \end{pmatrix} = \begin{pmatrix} q^m & q^{m/2} & q^m \\ q^{m/2} & 1 & q^{m/2} \\ q^m & q^{m/2} & 1 \end{pmatrix} \begin{pmatrix} X_{m-1}(a|0,L-2) \\ X_{m-1}(a|L-2,L-2) \\ X_{m-1}(a|L-4,L-2) \end{pmatrix}. \tag{3.14}$$

The initial condition is $X_0(a|b,c) = \delta_{a,b}$ if $c = b, b \pm 2$. We compute X_m up to any finite m using this recursion relation (3.14) and obtain the lower order terms for X_∞. In fact up to order $m/2$ the terms in $X_m(a|b,c)$ and those in $\lim_{m\,\mathrm{even} \to \infty} X_m(a|b,c)$ are exactly the same.

4. Results of the LSP for the spin-1 CSOS model

4.1. *The case of odd L*

The solutions of the recursion relation (3.14) for finite m may have cumbersome expressions. For the thermodynamic limit we can expect that they will have more beautiful expressions. Indeed they do. One observation is that when we compute X_m using the recursion relation, more and more large coefficients appear with respect to terms with large order of q. But if we multiply X_m by $(\phi(q))^2 = (\prod_{n=1}^{\infty}(1-q^n))^2$, there remain only small coefficients for any order of q. By this observation we have been able to find that $X_\infty \cdot (\phi(q))^2$ should certainly be expressed by theta functions. The following results are based on this type of computational work.

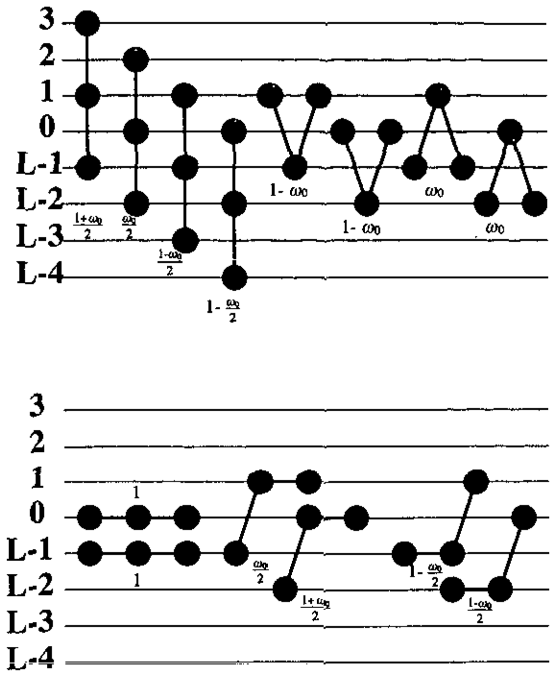

Figure 1. Exceptional values of the weight function $H(\bullet,\bullet,\bullet)$ of the L-state spin 1 (2-fusion) CSOS model.

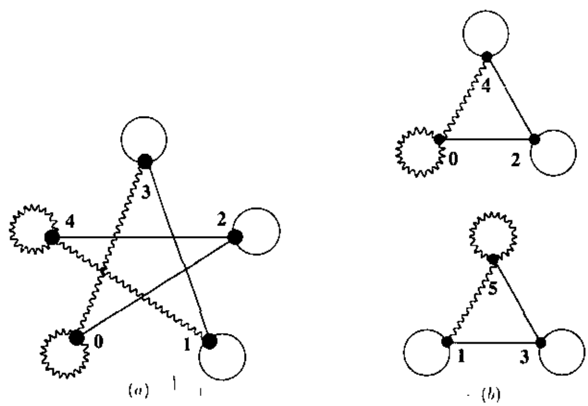

Figure 2. Incidence diagrams for 2-fusion CSOS models. Each pair of state variables directly connected with either a full line or wavy line is admissible on nearest-neighbour lattice sites. The pair with a wavy line is not allowed for a background configuration: (*a*) 5-state model; (*b*) 6-state model.

We write the thermodynamic limit of the 1D configuration sum as

$$\lim_{m\,\mathrm{even}\to\infty} X_m(a|b,c) = q^{(b-a)/4 - r^2/4(L-2) - s^2/16 + 1/8 + a^2/4L}\, c_{r,s,a}(q) \tag{4.1}$$

where $r \equiv (b+c-2)/2$ and $s \equiv (b-c+2)/2+1$. The extra fractional power of q is introduced for the convenience of later analyses. We call $c_{r,s,a}(q)$ the branching coefficient. We find its explicit form for the spin-1 CSOS model using the calculation method for X_m referred to in the previous section. At first we shall give the result for the $L(=\text{odd})$-state spin-1 CSOS model. Its branching coefficient is expressed as

$$c_{r,s,a}(q) = \frac{q^{-1/16}}{2\phi(q)^2}\left\{\phi(-q^{1/2}) \sum_{n=-\infty}^{\infty} q^{[L(L-2)/2](n+J/L(L-2))^2} + \phi(q^{1/2}) \sum_{n=-\infty}^{\infty} (-)^n q^{[L(L-2)/2](n+J/L(L-2))^2}\right\} \tag{4.2}$$

for $s = 1, 3$ where $J = 0, 1, 2, \ldots, L(L-2)$ and

$$c_{r,s,a}(q) = \frac{\phi(q^2)}{\phi(q)^2} \sum_{n=-\infty}^{\infty} q^{[L(L-2)/2](n+J/L(L-2))^2} \tag{4.3}$$

for $s = 2$ where $J = 1/2, 3/2, \ldots, L(L-2)/2$. The values of $J = J(r,s,a)$ are determined by a definite rule. Let us explain the rule by the model with $L = 5$ (figure 2(a)) as an example. The corresponding values of J are listed in tables 1 and 2. At first let $J(0,1,0) = 0$ and $J(1,2,0) = L/2$. Next fill the tables diagonally from the upper left to the lower right with these integers or half odd integers increasingly and then decreasingly. The top and bottom sides of each of the tables are identified. The left- and right-hand sides are also identified when $s = 2$, and identified with a 'twist' with respect to s when $s = 1, 3$.

Table 1. The values of J for $L = 5$ model for $s = 1, 3$.

$(b,c) = \{$	(0,2) / (2,0)	(1,3) / (3,1)	(2,4) / (4,2)
$(r$	0	1	2)
$a = 0$	0 / 15	5 / 10	10 / 5
$a = 1$	6 / 9	1 / 14	4 / 11
$a = 2$	12 / 3	7 / 8	2 / 13
$a = 3$	12 / 3	13 / 2	8 / 7
$a = 4$	6 / 9	11 / 4	14 / 1

As seen earlier, this branching coefficient is neatly expressed by modular functions. We can convert it into another form in order to analyse its behaviour at $q = e^{-8\pi^2/\epsilon} \to 1$. Indeed this can only be done by one technical, mathematical procedure of the modular transformation (see (A.7)–(A.11) in the appendix). In terms of

$t = e^{-\varepsilon/2}$, it is

$$c_{r,s,a}(q) = \frac{1}{2\phi(t)\sqrt{L(L-2)}}\left[t^{-1/16}\left(\prod_{n=0}^{\infty}(1+t^{n+1/2})\right)\right.$$
$$\times\left(1+2\sum_{n=1}^{\infty}t^{n^2/2L(L-2)}\cos\frac{2n\pi J}{L(L-2)}\right)$$
$$\left.+\sqrt{2}\left(\prod_{n=0}^{\infty}(1+t^n)\right)\sum_{n=0}^{\infty}t^{(n+1/2)^2/2L(L-2)}\cos\frac{(2n+1)\pi J}{L(L-2)}\right] \qquad (4.4)$$

when $s = 1, 3$ and

$$c_{r,s,a}(q) = \frac{t^{-1/16}}{\sqrt{2L(L-2)}}\frac{\phi(t^2)}{\phi(t)^2}\left(1+2\sum_{n=1}^{\infty}t^{n^2/2L(L-2)}\cos\frac{2n\pi J}{L(L-2)}\right) \qquad (4.5)$$

when $s = 2$.

Table 2. The values of J for $L = 5$ model for $s = 2$.

(b, c) $(r$	$(1, 1)$ 0	$(2, 2)$ 1	$(3, 3)$ 2)
$a = 0$	15/2	5/2	5/2
$a = 1$	3/2	13/2	7/2
$a = 2$	9/2	1/2	11/2
$a = 3$	9/2	11/2	1/2
$a = 4$	3/2	7/2	13/2

4.2. *The case of even L*

Next we give the result for the $L(=\text{even})$-state spin-1 CSOS model. The branching coefficient is

$$c_{r,s,a}(q) = \frac{q^{-1/16}}{2\phi(q)^2}\left\{\phi(-q^{1/2})\sum_{n=-\infty}^{\infty}q^{L(L-2)/8)(n+2J/L(L-2))^2}\right.$$
$$\left.+\phi(q^{1/2})\sum_{n=-\infty}^{\infty}(-)^n q^{(L(L-2)/8)(n+2J/L(L-2))^2}\right\} \qquad (4.6)$$

for $s = 1, 3$ where $J = 0, 1, 2, \ldots, L(L-2)/2$ and

$$c_{r,s,a}(q) = \frac{\phi(q^2)}{\phi(q)^2}\sum_{n=-\infty}^{\infty}q^{(L(L-2)/8)(n+2J/L(L-2))^2} \qquad (4.7)$$

Table 3. The values of J for $L=6$ model for $s=1,3$.

$(b,c)=\{$	(0,2) (2,0)	(1,3) (3,1)	(2,4) (4,2)	(3,5) (5,3)
(r	0	1	2	3)
$a=0$	0 12		6 6	
$a=1$		1 11		5 7
$a=2$	8 4		2 10	
$a=3$		9 3		3 9
$a=4$	8 4		10 2	
$a=5$		7 5		11 1

Table 4. The values of J for $L=6$ model for $s=2$.

(b,c)	(1,1)	(2,2)	(3,3)	(4,4)
(r	0	1	2	3)
$a=0$		3		3
$a=1$	2		4	
$a=2$		1		5
$a=3$	6		0	
$a=4$		5		1
$a=5$	2		4	

for $s=2$ where $J=0,1,2,\ldots,L(L-2)/4$. The values of $J=J(r,s,a)$ are determined by the same rule as for the $L(=\text{odd})$ state model. Here we shall present the $L=6$ model (figure 2(*b*)), as an example, in tables 3 and 4.

In terms of t, it results in

$$
\begin{aligned}
c_{r,s,a}(q) = \frac{1}{\phi(t)\sqrt{L(L-2)}} \Bigg[& t^{-1/16}\left(\prod_{n=0}^{\infty}(1+t^{n+1/2})\right) \\
& \times \left(1+2\sum_{n=1}^{\infty} t^{2n^2/L(L-2)} \cos\frac{4n\pi J}{L(L-2)}\right) \\
& + \sqrt{2}\left(\prod_{n=0}^{\infty}(1+t^n)\right)\sum_{n=0}^{\infty} t^{2(n+1/2)^2/L(L-2)} \cos\frac{2(2n+1)\pi J}{L(L-2)}\Bigg]
\end{aligned}
\tag{4.8}
$$

when $s=1,3$ and

$$
c_{r,s,a}(q) = \frac{2t^{-1/16}}{\sqrt{2L(L-2)}}\frac{\phi(t^2)}{\phi(t)^2}\left(1+2\sum_{n=1}^{\infty} t^{2n^2/L(L-2)}\cos\frac{4n\pi J}{L(L-2)}\right) \tag{4.9}
$$

when $s=2$.

4.3. Result of the LSP for general L

In both of the above two (odd and even L) cases, the $c_{r,s,a}(q)$ can be expressed in terms of the branching coefficients defined through the branching rule between the elliptic theta functions,

$$\Theta^{(\varepsilon_1,\varepsilon_2)}_{j_1,m_1}(z,q)\frac{\Theta^{(-,+)}_{j_2,m_2}(z,q)}{\Theta^{(-,+)}_{1,2}(z,q)}=\sum_{j_3=0}^{m_3}c^{(\varepsilon_1,\varepsilon_2)}_{j_1,j_2,j_3}(q)\Theta^{(\varepsilon_1,\varepsilon_2)}_{j_3,m_3}(z,q) \tag{4.10}$$

$$\Theta^{(\varepsilon_1,\varepsilon_2)}_{j,m}(z,q)\equiv\sum_{\nu\in\mathbf{Z},\gamma=\nu+j/2m}(\varepsilon_2)^\nu q^{m\gamma^2}(z^{-m\gamma}+\varepsilon_1 z^{m\gamma}) \tag{4.11}$$

where $m_3=m_1+m_2-2$ and $\varepsilon_1,\varepsilon_2=\pm1$. These branching coefficients $c^{(\varepsilon_1,\varepsilon_2)}_{j_1,j_2,j_3}(q)$ depend on m_1 and m_2, although we are suppressing these labels. With the explicit expressions for $m_2=4$ listed in the appendix ((A.5) and (A.6)) one can identify

$$c_{r,s,a}(q)=\tfrac{1}{2}\left[(1+\delta_{a,0})\left(c^{(+,+)}_{r,s,a}(q)+c^{(+,+)}_{r,s,L-a}(q)\right)+c^{(-,+)}_{r,s,a}(q)-c^{(-,+)}_{r,s,L-a}(q)\right] \tag{4.12}$$

where the branching coefficients on the right-hand side are those with $m_1=L-2$ and $m_2=4$.

Let us consider the normalization of the LSP. When multiplying four corner transfer matrices, two of them are $\pi/2$-rotated by replacing the spectral parameter u by $-1-u$ and with an additional gauge transformation (3.4). In addition when we take the double limit (2.1) the Boltzmann weight has been gauge transformed. Most of these gauge factors will cancel one another out between the edges of the neighbouring plaquettes, but those at the 'boundary' and central sites survive. The former are irrelevant when the LSP is considered, because we take a ratio of the partial traces in order to get the LSP. The latter is composed of $G_a(u)^2\times G_a(-1-u)^2=x^{-a(a-L)/2L}$ and

$$g_a^2=\left(\varepsilon_a\sqrt{\Theta(a)}\right)^2=\Theta(a)=x^{L/8}\left(\frac{2\pi L}{\varepsilon}\right)^{1/2}\Theta^{(+,+)}_{a,L}(x,x^2). \tag{4.13}$$

Multiplying these gauge factors to the 1D sum (4.1) and neglecting a-independent factors we get the partial trace as

$$\mu(a|b,c)=\Theta^{(+,+)}_{a,L}(x,x^2)\cdot c_{r,s,a}(x^2). \tag{4.14}$$

For the normalization of the LSP we take the summation of $\mu(a|b,c)$ with respect to a. At first we can see that

$$\begin{aligned}\sum_{a=0}^{L-1}&\left(c^{(-,+)}_{r,s,a}(x^2)-c^{(-,+)}_{r,s,L-a}(x^2)\right)\Theta^{(+,+)}_{a,L}(x,x^2)\\ &=\sum_{a=0}^{L-1}c^{(-,+)}_{r,s,a}(x^2)\Theta^{(+,+)}_{a,L}(x,x^2)-\sum_{a=0}^{L-1}c^{(-,+)}_{r,s,L-a}(x^2)\Theta^{(+,+)}_{L-a,L}(x,x^2)\\ &=c^{(-,+)}_{r,s,0}(x^2)\Theta^{(+,+)}_{0,L}(x,x^2)-c^{(-,+)}_{r,s,L}(x^2)\Theta^{(+,+)}_{L,L}(x,x^2)\\ &=0\end{aligned} \tag{4.15}$$

because $\Theta^{(+,+)}_{L-a,L}(x,x^2) = \Theta^{(+,+)}_{a,L}(x,x^2)$ and $c^{(-,+)}_{r,s,0}(q) = c^{(-,+)}_{r,s,L}(q) = 0$. Next we can prove that

$$\sum_{a=0}^{L-1}(1+\delta_{a,0})\left(c^{(+,+)}_{r,s,a}(x^2) + c^{(+,+)}_{r,s,L-a}(x^2)\right)\Theta^{(+,+)}_{a,L}(x,x^2)$$
$$= \sum_{a=0}^{L-1}(1+\delta_{a,0})c^{(+,+)}_{r,s,a}(x^2)\Theta^{(+,+)}_{a,L}(x,x^2)$$
$$+ \sum_{a=0}^{L-1}(1+\delta_{a,0})c^{(+,+)}_{r,s,L-a}(x^2)\Theta^{(+,+)}_{L-a,L}(x,x^2)$$
$$= \sum_{a=0}^{L} c^{(+,+)}_{r,s,a}(x^2)\Theta^{(+,+)}_{a,L}(x,x^2) + \sum_{a=0}^{L} c^{(+,+)}_{r,s,L-a}(x^2)\Theta^{(+,+)}_{L-a,L}(x,x^2)$$
$$= 2\times\Theta^{(+,+)}_{r,L-2}(x,x^2)\times\frac{\Theta^{(-,+)}_{s,4}(x,x^2)}{\Theta^{(-,+)}_{1,2}(x,x^2)} \tag{4.16}$$

using the theta-function branching rule (4.10). The identity (4.12) shows that we can put the previous two sums together and get

$$\sum_a \mu(a|b,c) = \frac{\Theta^{(+,+)}_{r,L-2}(x,x^2)\Theta^{(-,+)}_{s,4}(x,x^2)}{\Theta^{(-,+)}_{1,2}(x,x^2)}. \tag{4.17}$$

Therefore the result of the LSP for the 2-fusion CSOS model is given by

$$P(a|b,c) = \frac{\Theta^{(+,+)}_{a,L}(x,x^2)\Theta^{(-,+)}_{1,2}(x,x^2)}{\Theta^{(+,+)}_{r,L-2}(x,x^2)\Theta^{(-,+)}_{s,4}(x,x^2)}c_{r,s,a}(x^2) \tag{4.18}$$

with branching coefficient (4.12). Formula (4.18) is the main result of this paper.

We can write it in a more suitable form for analysing its critical behaviour as follows

$$P(a|b,c) = \sqrt{\frac{2(L-2)}{L}}\frac{\vartheta_4(\pi a/L, t^{2/L})\vartheta_1(\frac{1}{2}\pi, t)}{\vartheta_4(\pi r/(L-2), t^{2/(L-2)})\vartheta_1(\frac{1}{4}\pi s, t^{1/2})}c_{r,s,a}(x^2). \tag{4.19}$$

Putting this together with (4.4) and (4.5) ((4.8) and (4.9)), the branching coefficient's expressions in terms of t, we can see the leading terms from the critical point: when L is odd

$$P(a|b,c) = \frac{1}{L}\left(1 + 2t^{1/2L(L-2)}\cos\frac{2\pi J}{L(L-2)} + \cdots\right) \tag{4.20}$$

and when L is even

$$P(a|b,c) = \frac{2}{L}\left(1 + 2t^{2/L(L-2)}\cos\frac{4\pi J}{L(L-2)} + \cdots\right). \tag{4.21}$$

5. Concluding remarks

In this paper we have obtained the LSP for the spin-1 ($N = 2$) CSOS model. We take the double limit (low temperature and extreme anisotropy limit) of its Boltzmann weights, obtain the weight function $H(\bullet,\bullet,\bullet)$ and the recursion relation for the 1D configuration sum $X_m(a|b,c)$. We see that the thermodynamic limit of the 1D sum is expressed by branching coefficients. Using theta-function identities we finally arrive at an exact form for the LSP (4.18).

In [5] the LSP of the spin-1/2 CSOS model (regime III of their $A^{(1)}_{L-1}$ model) has already been given. Comparing it with that of spin-1 CSOS model in this paper, we can recognize a systematic extension of these models in terms of the LSP. The LSP of the spin-1/2 CSOS model is related to the theta-function branching rule (4.10) with $m_1 = L - 1$ and $m_2 = 3$, while that of the spin-1 CSOS model is related to the rule with $m_1 = L - 2$ and $m_2 = 4$. The LSPs of both models formally have the same combinations of the four branching coefficients as given in (4.12). Although the fusion construction is a combinatorial procedure performed on the local Boltzmann weights of the model, the resultant LSP seems to admit a systematic, mathematical extension.

We already know examples which have a similar mathematical structure that, in fact, governs hierarchies of elliptic solutions of the Yang–Baxter equation, and therefore hierarchies of off-critical lattice models. As an example, and for a comparison with our models, let us see what is known about the hierarchy of those RSOS models in [3]. For the spin-$N/2$ RSOS model the LSP is

$$P(a|b,c) = \frac{\Theta^{(-,+)}_{a,L}(x,x^2)\Theta^{(-,+)}_{1,2}(x,x^2)}{\Theta^{(-,+)}_{r,L-N}(x,x^2)\Theta^{(-,+)}_{s,N+2}(x,x^2)} c^{(-,+)}_{r,s,a}(x^2) \tag{5.1}$$

where $r \equiv (b + c - N)/2$, $s \equiv (b - c + N)/2 + 1$, and $c^{(-,+)}_{r,s,a}(x^2)$ is that given in (4.10) specified with $m_1 = L - N$ and $m_2 = N + 2$.

Having been inspired by the resultant LSP forms of the unfused CSOS model, the 2-fusion CSOS model and the result of the N-fusion RSOS model, we may conjecture on the form of the spin-$N/2$ CSOS model. That is

$$P(a|b,c) = \frac{\Theta^{(+,+)}_{a,L}(x,x^2)\Theta^{(-,+)}_{1,2}(x,x^2)}{\Theta^{(+,+)}_{r,L-N}(x,x^2)\Theta^{(-,+)}_{s,N+2}(x,x^2)} c_{r,s,a}(x^2) \tag{5.2}$$

where $c_{r,s,a}(x^2)$ is defined by the same combination as in (4.12), and specified with $m_1 = L - N$ and $m_2 = N + 2$.

There are some possibly substantial differences between (5.1) and (5.2). Two theta functions in the first factor are replaced by those of another kind, and the branching coefficients are different. The difference with respect to the branching coefficients is more substantial, since they cannot be obtained unless a full calculation of the 1D sum has been completed. When one sees the definitions of the branching coefficients such as (4.12) specified with $m_1 = L - N$ and $m_2 = N + 2$, at first one may probably think that $c_{r,s,a}(q)$ is more complicated than $c^{(-,+)}_{r,s,a}(q)$. However, as seen in this paper for the 2-fusion case, $c_{r,s,a}(q)$ has a somewhat simpler form.

We shall give a naïve analysis of the criticality of these models in the way of [2]. We refer the critical behaviour of the L-state spin-$N/2$ RSOS model,

$$P(a|b,c) = P_c(a) \cdot (1 + \bullet \cdot t^{\Delta} + \cdots) \tag{5.3}$$

where

$$P_c(a) = (4/L)\sin^2(\pi a/L) \tag{5.4}$$

is the LSP for the critical model, and

$$\Delta = 3N/4L(L-N). \tag{5.5}$$

The nome t indicates the deviation from the critical point of the model. The factor '•' which depends on a, b, c is the leading contribution from the boundary of the system when the temperature is slightly lowered from the T_c. If conjecture (5.2) is true our fusion CSOS model also has such a critical property,

$$P(a|b,c) = P_c \cdot \left(1 + \bullet \cdot t^{\Delta} + \cdots\right). \tag{5.6}$$

In this case the LSP for the critical model is

$$P_c = \begin{cases} 1/L & \text{for odd } L \\ 2/L & \text{for even } L \end{cases} \tag{5.7}$$

which is consistent with the fact that a critical fusion CSOS model is equal to a vertex model and P should not depend on a there. And from that conjecture the exponent is found to be

$$\Delta = \begin{cases} N/4L(L-N) & \text{for odd } L \\ N/L(L-N) & \text{for even } L. \end{cases} \tag{5.8}$$

According to the results in this paper this is correct for $N = 2$ ((4.20) and (4.21)). This is also correct for $N = 1$ as was shown in [5]. Whether this (5.8) is also valid for higher ($N \geqslant 3$) fusion CSOS models or not needs further investigation. Studies on examining its validity are now in progress, while the calculation of LSP in other regimes is another remaining problem.

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Appendix

A1. Explicit form of the branching coefficients

$$\Theta^{(\pm)}_{a,b}(q) \equiv q^{a^2/2b} \sum_{n=-\infty}^{\infty} (\pm)^n q^{(b/2)n^2 + an} \tag{A.1}$$

$$\varepsilon^a_b \equiv \begin{cases} \frac{1}{2} & \text{for } b \equiv 0 \bmod a \\ 1 & \text{otherwise} \end{cases} \tag{A.2}$$

$$k \equiv j_1 m_3 - j_3 m_1 \qquad l \equiv j_1 m_3 + j_3 m_1 \qquad n \equiv m_1 m_3. \tag{A.3}$$

(a) $m_2 = 3$ $(j_3 \equiv j_1 + j_2 - 1 \bmod 2)$

$$c^{(\pm,+)}_{j_1,j_2,j_3}(q) = \varepsilon^{m_3}_{j_3} \frac{q^{-1/24}}{\phi(q)} \left(\Theta^{(+)}_{k,2n}(q) \pm \Theta^{(+)}_{l,2n}(q) \right) \tag{A.4}$$

(b) $m_2 = 4$ $(j_3 \equiv j_1 + j_2 - 1 \bmod 2)$
$j_2 = 1, 3$

$$c^{(\pm,+)}_{j_1,j_2,j_3}(q) = \varepsilon^{m_3}_{j_3} \frac{q^{-1/16}}{2\phi(q)^2} \Big[\phi(-q^{1/2}) \left(\Theta^{(+)}_{k/2,n}(q) \pm \Theta^{(+)}_{l/2,n}(q) \right) + (-)^{(j_1+j_2-1-j_3)/2} \phi(q^{1/2}) \left(\Theta^{((-)^{m_1})}_{k/2,n}(q) \pm (-)^{j_1} \Theta^{((-)^{m_1})}_{l/2,n}(q) \right) \Big] \tag{A.5}$$

$j_2 = 2$

$$c^{(\pm,+)}_{j_1,j_2,j_3}(q) = \varepsilon^{m_3}_{j_3} \frac{\phi(q^2)}{\phi(q)^2} \left(\Theta^{(+)}_{k/2,n}(q) \pm \Theta^{(+)}_{l/2,n}(q) \right). \tag{A.6}$$

A2. Identities between modular functions expressed in terms of conjugate nomes q and t

$$q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sqrt{\frac{\varepsilon}{4\pi}} t^{1/24} \prod_{n=1}^{\infty} (1 - t^n) \tag{A.7}$$

$$q^{-1/48} \prod_{n=0}^{\infty} (1 + q^{n+1/2}) = t^{-1/48} \prod_{n=0}^{\infty} (1 + t^{n+1/2}) \tag{A.8}$$

$$q^{-1/48} \prod_{n=0}^{\infty} (1 - q^{n+1/2}) = \frac{1}{\sqrt{2}} t^{1/24} \prod_{n=0}^{\infty} (1 + t^n) \tag{A.9}$$

$$\sum_{n=-\infty}^{\infty} q^{(b/2)(n+a/b)^2} = \sqrt{\frac{\varepsilon}{4\pi b}} \left(1 + 2 \left(\sum_{n=1}^{\infty} t^{n^2/2b} \cos\left[2n\pi \frac{a}{b} \right] \right) \right) \tag{A.10}$$

$$\sum_{n=-\infty}^{\infty} (-)^n q^{(b/2)(n+a/b)^2} = \sqrt{\frac{\varepsilon}{4\pi b}} 2 \left(\sum_{n=0}^{\infty} t^{(n+1/2)^2/2b} \cos\left[(2n+1)\pi \frac{a}{b} \right] \right). \tag{A.11}$$

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